A Study of Anyon Statistics by Breit Hamiltonian Formalism

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ABSTRACT

We study the anyon statistics of a $2+1$ dimensional Maxwell-Chern-Simons (MCS) gauge theory by using a systemmetic metheod, the Breit Hamiltonian formalism.

1 Introduction

In fundamental physics, it is important to classify elementary particles by their quantum characteristics, espacially the spin of particles. Phenomenally and theoretically, in the space-time dimensions equal or greater than $3+1$ dimension, there are just fermions and bosons with half-integer and integer spin respectively. In two spatial dimensions, since the universal covering group, $SO(2)$, is noncompact and the angular momentum algebra is a commutation algebra, the elementary particles get arbitrary values of spin and should obey fractional statistics. These particles are called *anyons* [1][2].

In the usual prescriptions of $2+1$ dimensional quantun field theory, one may impose a magnetic fluxtube upon every charged particle (charge-fluxtube composites) [3][4], and then, a phase factor, $e^{i\theta}$, will appears in an anyon wavefunction ,while two identical particles are made to encircle each other. The θ is a constant phase angle, called the anyon phase angle. This phase could be treated in terms of wilson loops, and it is closely related to the Aharonov-Bohm(AB) effect [5][6]. Viewed in this way, the gauge theory may be approximatelly described by the non-dynamical Chern-Simons action rather than the Maxwell's one [3]. But, in the true charge-fluxtube composites, the Coulomb interaction will cause complicated situations. Therefore, in this paper, we would like to treat the statistical phase as a AB effect by imposing a magnetic momentum, $\mathbf{m}$, on every charged particle, but avoiding the assumption of Charge-fluxtube composites [7][8].

The crucial mission of the magnetic momentum is to redescribe the statistical interaction through an introduction of a $\mathbf{m} \cdot \mathbf{B}$ type interaction. The magnetic field $\mathbf{B}$ is aroused by the other moving charged particles. One may assume that the field is a extension version of $3+1$ dimensional Biot-Savart's law, and it will be the following form, in Heaviside-Lorentz units,

$$\begin{aligned} \mathbf{B} &= -\frac{q}{2\pi c^2} \frac{(\dot{\vec{\varrho}}_2 - \dot{\vec{\varrho}}_1) \times (\vec{\varrho}_2 - \vec{\varrho}_1)}{|\vec{\varrho}_2 - \vec{\varrho}_1|^2} \\ &= -\frac{q}{2\pi c^2} \frac{d}{dt}(\phi_{12}) = -\frac{q}{2\pi c^2} \frac{d}{dt}(\phi_{21}) \quad , \end{aligned} \tag{1}$$

where $\vec{\varrho}_1$ and $\vec{\varrho}_2$ are the position vectors of particle 1 and 2 with charge q respectively, and ϕ_{ab} is a relative azimuthal angle between two particles. Hence, the $\mathbf{m} \cdot \mathbf{B}$ interaction indeed become a statistical interaction, and the statistical potential is

$$L_\theta = -2\mathbf{m} \cdot \mathbf{B} = \left[\frac{q\mathbf{m}}{c^2}\right] \frac{1}{\pi} \dot{\phi} \quad . \tag{2}$$

The factor in the bracket is the anyon phase angle θ. Consequently, the anyon phase can be expressed as

$$exp\left[\frac{i}{\hbar} \int L_\theta dt\right] = exp\left[i\frac{\theta}{\pi}\delta\phi\right] \quad . \tag{3}$$

On our approach, the $\mathbf{m} \cdot \mathbf{B}$ interaction will be introduced naturally by identifying this interaction with spin-orbit term appears in Breit Hamiltonian. The detail will be discussed later.

We arrange this paper as follow: In next section, we illustrate what the Breit Hamiltonian is, and define some conventions of this paper. In the third section, the Breit Hamiltonian are calculated and analyzed in Coulomb gauge for three theories, *i.e.* the Maxwell, Chern-Simons and MCS gauge theories. Here, we choose a minimal coupling scheme for gauge field and Dirac fermions. It is also shown how one can naturally draw a $\mathbf{m} \cdot \mathbf{B}$ interaction into, once it is identified as a spin-orbit interaction. The last section, an electron-positron system is explained briefly. In the appendix, some useful Fourier transformations are exhibited.

2 Breit Hamiltonian:Non-Relativistic Limit of Dirac Theory

In some situations, physicists prefer to deal a quantum system in non-relativistic limit. This corresponds to a higher order correction of a potential belongs to Schrödinger equation. For instance, for an $U(1)$ gauge theory with Dirac fermions, the corrected potetial, up to the order of $\frac{\hbar^2}{c^2}$, is just the Breit Hamiltonian [9].

The starting point is to solve the Dirac equation in the non-relativistic limit with next term in the expansion of the relativistic expression for the kinetic energy. It leads to a Schrödinger-like equation and a Schrödinger wave function of a free particle φ_{Sch} should satisfy the following equation

$$H^0\varphi_{Sch} = (E - mc^2)\varphi_{Sch} \quad ,$$

$$H^0 = \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3c^2} \quad , \tag{4}$$

where H^0 is a free fermion Hamiltonian, E is the total energy and m is fermion mass. We will denote a spinor amplitude of a plane wave by ω. It is subject to normalization condition

$$\omega^*\omega = 1 \quad .$$

Therefore, the "bispinor" amplitude of a free particle in a plane then can be expressed in terms of ω, with correction, by

$$u(p) = \sqrt{2mc^2}\begin{pmatrix} (1 - \frac{\vec{p}^2}{8m^2c^2})\omega \\ \frac{\vec{\alpha}\cdot\vec{p}}{2mc}\omega \end{pmatrix} \quad , \tag{5}$$

where $\vec{\alpha} \equiv \hat{i} + \sqrt{-1}\hat{j}$.

We consider the minimal coupling theory of abelian gauge field and Dirac fermion described by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_A(A_\mu, \partial_\mu A_\nu) + \overline{\psi}(i\not{\partial} - \frac{e}{c}\not{A} - m)\psi \quad , \tag{6}$$

in which $\mathcal{L}_A(A_\mu, \partial_\mu A_\nu)$ associates with the behaviors of free photons and ψ is the fermion field. With the help of Eqs.(5) and (6), a tree level electron-electron scattering amplitude is

$$M_{fi} = e^2(\bar{u}'_1(p'_1)\gamma^\mu u_1(p_1))D_{\mu\nu}(q)(\bar{u}'_2(p'_2)\gamma^\mu u_2(p_2)) \quad , \tag{7}$$

where $p_1,\ p'_1,\ p_2,\ p'_2$ and transfer momentum $q = p'_1 - p_1 = p_2 - p'_2$ are arranged as Fig.1, $\vec{p} = 1/2\cdot(\vec{p}_1 - \vec{p}_2)$ and $D_{\mu\nu}$ is the photon propagator. Note that the ω_i is the spinor amplitude of particle i.

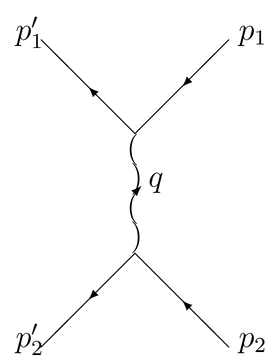

Fig 1. The tree diagram of e^-e^- scattering.

In non-relativistic limit, we define the amplitude

$$M_{fi} \equiv -4m^2c^4({\omega'_1}^*{\omega'_2}^* U(\vec{p},\vec{q})\omega_1\omega_2) \quad . \tag{8}$$

The $U(\vec{p},\vec{q})$ is an interacting potential. By a appropriately Fourier transformation,

$$\int e^{i\vec{q}\cdot\vec{\varrho}} U(\vec{p},\vec{q}) \frac{d^2\vec{q}}{(2\pi)^2} \quad , \tag{9}$$

where $\vec{\varrho} = \vec{\varrho}_1 - \vec{\varrho}_2$, the interaction potential $U(\vec{p},\vec{q})$ in equation (8) is precisely turned out to be the Breit Hamiltonian in coordinate space which describes the interaction between two indentical fermions non-relativistically.

In $2+1$ dimension, we defined the complete set of gamma matrice as

$$\gamma^0 \equiv \sigma^3 \quad , \quad \gamma^1 \equiv \sigma^3\sigma^1 \quad , \quad \gamma^2 \equiv \sigma^3\sigma^2 \quad . \tag{10}$$

The σ^i are usual Pauli matrice, they satisfy the following algebra

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\alpha}\gamma_\alpha \quad . \tag{11}$$

As space-time indices, all Greek letters take values from 0 to 2, and all latin letters take values from 1 to 2. Thereby, from Eqs.(5) and (10), we have some useful formulae which are needed as scattering amplitude is handled and are exhibited as :

$$\bar{u}'_1(p'_1)\gamma^0 u_1(p_1) = 2mc^2{\omega'_1}^* \left[1 - \frac{\vec{q}^2}{8m^2c^2} + \frac{i\vec{q}\times\vec{p}_1}{4m^2c^2}\right]\omega_1 \quad ,$$

$$\bar{u}'_1(p'_1)\vec{\gamma} u_1(p_1) = c^2(\frac{1}{c}){\omega'_1}^* \left[(2\vec{p}_1 + \vec{q}) + i((\vec{q})_x\hat{j} - (\vec{q})_y\hat{i})\right]\omega_1 \quad ,$$

$$\bar{u}'_2(p'_2)\gamma^0 u_2(p_2) = 2mc^2\omega'^*_2\left[1 - \frac{\vec{q}^2}{8m^2c^2} - \frac{i\vec{q}\times\vec{p}_2}{4m^2c^2}\right]\omega_2 \quad ,$$

$$\bar{u}'_2(p'_2)\vec{\gamma}u_2(p_2) = c^2(\frac{1}{c})\omega'^*_2\left[(2\vec{p}_2 - \vec{q}) - i((\vec{q})_x\hat{j} - (\vec{q})_y\hat{i})\right]\omega_2 \quad . \qquad (12)$$

We have used the conventions of vector, e.g. p_μ expressed a 3-vector and $(\vec{q})_x$ and $(\vec{q})_y$ are the x and y coponents of 2-vector $\vec{q}$ respectively. Throughout this paper we use the conventions $\epsilon^{012} = 1$ and Minkowski metric $g_{\mu\nu} = diag(+1,-1,-1)$. All of the amplitudes and Breit Hamiltonians of this paper are expressed in terms of centre-of-mass frame.

3 Anyon Phase Angles

Our main purpose is to analyze the anyon phase for a $2+1$ dimensional system with an abelian topologically massive Lagrangian density,

$$\mathcal{L}_A(A_\mu, \partial_\nu A_\mu) = \mathcal{L}_{MCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\alpha\beta\sigma}A_\alpha\partial_\beta A_\sigma \quad . \tag{13}$$

Before doing this, we first consider two simpler cases, Maxwell and CS gauge theory.

3.1 Maxwell Gauge Theory

In this case, the free photon Lagrangian density is set to be

$$\mathcal{L}_A(A_\mu, \partial_\nu A_\mu) = \mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad . \tag{14}$$

Now, the subsequent calculations are considerably simplified if the photon propagator $D_{\mu\nu}$ is chosen in the Coulomb gauge. The non-vanishing components of the propagator in $2+1$ dimension are

$$\begin{aligned} D_{00} &= -\frac{1}{\vec{q}^2} \quad , \\ D_{ij} &= \frac{1}{\vec{q}^2}(\delta_{ij} - \frac{(\vec{q})_i(\vec{q})_j}{\vec{q}^2}) \quad . \end{aligned} \tag{15}$$

Note that we take the non-relativistic approximation, $q^2 \simeq \vec{q}^2$, throughout this paper. The corresponding scattering amplitude, with the help of Eq.(12), is

$$M_{ij} = -4m^2c^4{\omega_1}'^*{\omega_2}'^* \left[e^2\frac{1}{\vec{q}^2} - \frac{e^2}{2m^2c^2} + i\frac{3e^2}{2m^2c^2}\frac{\vec{q}\times\vec{p}}{\vec{q}^2} \right.$$

$$\left. + \frac{e^2\vec{p}^2}{m^2c^2}\frac{1}{\vec{q}^2} - \frac{e^2}{m^2c^2}\frac{(\vec{q}\cdot\vec{p})^2}{\vec{q}^4} \right] \omega_1\omega_2 \quad . \tag{16}$$

This formula have been transformed into the type of centre-of-mass frame. It is clearly that particle interaction operator in momentum space version is just the function of $\vec{p}$ and $\vec{q}$ in the bracket of Eq.(16). Under Fourier transformations, which are displayed in appendix, one could obtain the following final expression for Breit Hamiltonian

$$U_M(\vec{p},\vec{\varrho}) = -\frac{e^2}{2\pi}\ln(\varrho) - \frac{3e^2\hbar}{4\pi m^2c^2}\frac{\vec{\varrho}\times\vec{p}}{\varrho^2} - \frac{e^2\vec{p}^2}{4\pi m^2c^2}\ln(\varrho)$$

$$+ \frac{e^2}{4\pi m^2c^2}\frac{1}{\varrho^2}\vec{\varrho}\cdot(\vec{\varrho}\cdot\vec{p})\vec{p} - \frac{e^2\hbar^2}{2m^2c^2}\delta^2(\vec{\varrho}) \quad , \tag{17}$$

where $\varrho = |\vec{\varrho}|$ and $\hbar$ have been put into appropriately.

The term proportional to $\vec{\varrho}\times\vec{p}$ in U_M represents spin-orbit interaction. Since $\vec{\varrho}\times\vec{p} = -\frac{1}{2}m\varrho^2\dot{\phi}$, the spin-orbit term, $-\frac{3e^2\hbar}{4\pi m^2c^2}\frac{\vec{\varrho}\times\vec{p}}{\varrho^2}$, is identified to $\mathbf{m}\cdot\mathbf{B}$ interaction. One could pick up an anyon phase angle from the terms like this. From Eq.(17), the anyon phase angle is

$$\theta_M = -\frac{3e^2}{8mc^2} \quad . \tag{18}$$

If the electron magnetic momentum, $\mathbf{m} = -\frac{q\hbar}{2mc}$, is put into Eq.(2), the statistical phase angle would be

$$\theta = -\frac{q^2}{2mc^2} \quad . \tag{19}$$

It disagrees with θ_M as $q^2 = e^2$. This is because that the purely kinematical effect, the Thomas precession effect, $-\frac{q^2}{8mc^2}$, is absent in the non-relativistic limit. In fact, these results had been calculated and analyzed by Hansson, Sporre and Leinass [7]. We showed this again just for a purpose that we could compare these outcomes with the results of $\mathcal{L}_{MCS}$.

3.2 Chern-Simons Gauge Theory

Recently a lot of attention has been given to CS gauge theory in two spatial dimensions [10][11][12][13]. Because it contributed remarkable features in different topics of physics, e.g. superconductivity, quantum hall effect, etc. That is why we like to study the Breit Hamiltonian in CS theory.

The CS Lagrangian density is defined as

$$\mathcal{L}_A(A_\mu, \partial_\nu A_\mu) = \mathcal{L}_{CS} = \frac{\mu}{2}\epsilon^{\alpha\beta\sigma} A_\alpha \partial_\beta A_\sigma \quad . \tag{20}$$

Two non-zero components of propagator in the Coulomb gauge are

$$\begin{aligned} D_{0i} &= \frac{i}{\mu}\epsilon_{0ij}(\vec{q})^j \frac{1}{\vec{q}^2} \quad , \\ D_{i0} &= -\frac{i}{\mu}\epsilon_{0ij}(\vec{q})^j \frac{1}{\vec{q}^2} \quad . \end{aligned} \tag{21}$$

Following the same procedures of section 3.1, the interacting potential is less complicated than the previous case. It is

$$U_{CS}(\vec{p}, \vec{q}) = -i\frac{2e^2\hbar}{\mu mc}\frac{\vec{q}\times\vec{p}}{\vec{q}^2} + \frac{e^2\hbar^2}{\mu mc} \quad . \tag{22}$$

By a Fourier transformation, the Breit Hamiltonian becomes

$$U_{CS}(\vec{p}, \vec{\varrho}) = \frac{e^2\hbar}{\pi\mu mc}\frac{1}{\varrho^2}\vec{\varrho} \times \vec{p} + \frac{e^2\hbar^2}{\mu mc}\delta^2(\vec{\varrho}) \quad . \tag{23}$$

In this case, coulomb-like potential is absent. The anyon phase angle is again given by the $\vec{\varrho} \times \vec{p}$ term, and reads

$$\theta_{CS} = \frac{e^2}{2\mu c} \quad . \tag{24}$$

This result is the same as the outcome derived by Haugset and Ravndal from a different starting point [14][15][16]. They got a effective partition function of CS theory by integrating out the degrees of freedom of the gauge field. In non-relativistic limit, the effective action equivalent to a statistical term,

$$exp\left[i\frac{\theta}{\pi}\delta\phi\right] \quad , \tag{25}$$

if the condition, $\theta = \frac{e^2}{2\mu c}$, is set up. This phase angle condition is just the Eq.(24).

If μ approaches to infinity, the Dirac fermions would interact with each other classically. It means that, these particles obey a distribution which is close to the Boltzmann type rather than the usual quantum statistics. Therefore, the statistical phase angle, θ_{CS}, is tended to zero as μ goes to infinity.

3.3 Abelian Topologically Massive Gauge Theory

A $2+1$ dimentional topologically massive gauge theory is governed by a photon Lagrangian density as

$$\mathcal{L}_A(A_\mu, \partial_\nu A_\mu) = \mathcal{L}_{MCS} = \mathcal{L}_M + \mathcal{L}_{CS}$$
$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\alpha\beta\sigma}A_\alpha\partial_\beta A_\sigma \quad . \tag{26}$$

Then the photon propagator, chosen in the Coulomb gauge, up to order $\frac{1}{c}$, reads

$$D_{00} = \frac{-1}{\vec{q}^2+\mu^2} + O(\frac{1}{c^2}) \quad ,$$
$$D_{0i} = i\mu\frac{1}{\vec{q}^2}\frac{1}{\vec{q}^2+\mu^2}\epsilon_{0ij}(\vec{q})^j + O(\frac{1}{c^2}) \quad ,$$
$$D_{i0} = -i\mu\frac{1}{\vec{q}^2}\frac{1}{\vec{q}^2+\mu^2}\epsilon_{0ij}(\vec{q})^j + O(\frac{1}{c^2}) \quad ,$$
$$D_{ij} = \frac{1}{\vec{q}^2+\mu^2}\delta_{ij} - \frac{1}{\vec{q}^2}\frac{(\vec{q})_i(\vec{q})_j}{\vec{q}^2+\mu^2} + O(\frac{1}{c}) \quad . \tag{27}$$

After some straightforward calculation, the interacting potential is

$$U_{MCS}(\vec{p},\vec{q}) = \frac{e^2}{\vec{q}^2+\mu^2}\left[1+\frac{\mu}{mc}+\frac{\vec{p}^2}{m^2c^2}\right] - \frac{e^2}{2m^2c^2}\frac{\vec{q}^2}{\vec{q}^2+\mu^2}$$
$$+\frac{i3e^2}{2m^2c^2}\frac{\vec{q}\times\vec{p}}{\vec{q}^2+\mu^2} - \frac{i2e^2\mu}{mc}\frac{\vec{q}\times\vec{p}}{\vec{q}^2(\vec{q}^2+\mu^2)}$$
$$-\frac{e^2}{m^2c^2}\frac{(\vec{q}\cdot\vec{p})^2}{\vec{q}^2(\vec{q}^2+\mu^2)} \quad , \tag{28}$$

where $\hbar$ is not put into yet. Note that the first order part of D_{ij}, which is denoted by $O(\frac{1}{c})$, is not contributed to U_{MCS}. This formula manifestly will go back to the results of sections 3.1 and 3.2 once one take the limit $|\mu| \to 0$ and $|\mu| \to \infty$ respectively.

Using those Fourier transformions in the appendix, the whole complicated Breit Hamiltonian is

$$
\begin{aligned}
U_{MCS}(\vec{p},\vec{\varrho}) = \frac{e^2}{2\pi}K_0(|\mu|\varrho) \\
+\frac{e^2}{2\pi}K_0(|\mu|\varrho)\left[\frac{\mu}{mc}+\frac{\mu^2}{2m^2c^2}+\frac{\vec{p}^2}{m^2c^2}-\frac{1}{m^2c^2}\frac{1}{\varrho^2}\vec{\varrho}\cdot(\vec{\varrho}\cdot\vec{p})\vec{p}\right] \\
+\frac{e^2}{2\pi}K_1(|\mu|\varrho)\frac{1}{\varrho}\left[-\frac{3\hbar|\mu|}{2m^2c^2}-\frac{\mu}{|\mu|}\frac{2\hbar}{mc}\right]\vec{\varrho}\times\vec{p}+\frac{e^2\hbar}{\pi\mu mc}\frac{1}{\varrho^2}\vec{\varrho}\times\vec{p} \\
+\frac{e^2\hbar^2}{2\pi|\mu|m^2c^2}K_1(|\mu|\varrho)\frac{1}{\varrho}\left[\frac{-2}{\varrho^2}\vec{\varrho}\cdot(\vec{\varrho}\cdot\vec{p})\vec{p}+\vec{p}^2\right] \\
+\frac{e^2\hbar^2}{2\pi\mu^2m^2c^2}\frac{1}{\varrho^2}\left[\frac{2}{\varrho^2}\vec{\varrho}\cdot(\vec{\varrho}\cdot\vec{p})\vec{p}-\vec{p}^2\right]+\delta^2(\vec{\varrho})\left[\frac{-e^2\hbar^2}{2m^2c^2}+\frac{e^2\hbar^2\vec{p}^2}{2\mu^2m^2c^2}\right] \quad , \qquad (29)
\end{aligned}
$$

where $K_i(|\mu|\varrho)$ is the modified Bessel function of the $i-$th rank.

There is something different from previous sections. Here, we should analyze two physics stages, the statistics and Breit Hamiltonian, while the extreme cases of ϱ are taken.

First, the statistics problem is discussed. The statistical Lagrangian one form is

$$
L_\theta dt = \left[-\frac{3e^2\hbar}{8mc^2}K_1(|\mu|\varrho)|\mu|\varrho-\frac{\mu}{|\mu|}\frac{e^2\hbar}{2c}K_1(\mu\varrho)\varrho+\frac{e^2\hbar}{2\mu c}\right]d\phi \quad , \qquad (30)
$$

which is read from $\vec{\varrho}\times\vec{p}$ terms. Since it is not an exact differential form, the anyon phase is mixed in the path integral of kernel,

$$
\mathbf{K}(\vec{\varrho}_f t_f,\vec{\varrho}_i t_i) \sim \int \mathcal{D}\left[\vec{\varrho}\right] exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}\left[L_\theta+L_0\right]dt\right] \quad , \qquad (31)
$$

where we have separated the whole Lagrangian into statistical one, L_θ, and the rest part, L_0. We denote i and f as initial and final state respectively. Obviously, the anyon phase can not be factored out as before.

However, it is interesting while a separation of the two particles is taken to be the limiting cases. When the separation goes to zero [17], the statistical Lagrangian one form is expressed as

$$\lim_{\varrho\to 0} L_\theta dt = -\frac{3e^2\hbar}{8mc^2} d\phi \quad . \tag{32}$$

Since it becomes an exact differential form, the anyon phase could be factored out again. And, it is just the case of Maxwell gauge theory. Conversely, the Lagrangian one form would be

$$\lim_{\varrho\to\infty} L_\theta dt = \frac{e^2\hbar}{2\mu c} d\phi \quad , \tag{33}$$

for $\varrho \to \infty$. This result falls into the case of the CS gauge theory. These two situations mean that the statistical effects of Maxwell action will dominate entire statistical behaviors as the separation of the two particles run to zero. And, when the two particles is separated far enough, the effect of CS action will cover the Maxwell's one.

Next, the asymptotic behaviors of Breit Hamiltonian are studied. For $\varrho \to 0$, the Breit Hamiltonian becomes

$$U_{MCS}(\vec{p},\vec{\varrho}) \simeq -\frac{e^2}{2\pi}\ln(\varrho)$$
$$-\frac{e^2}{2\pi}\ln(\varrho)\left[\frac{\mu}{mc} + \frac{\mu^2}{2m^2c^2} + \frac{\vec{p}^2}{m^2c^2} - \frac{1}{m^2c^2}\frac{1}{\varrho^2}\vec{\varrho}\cdot(\vec{\varrho}\cdot\vec{p})\vec{p}\right]$$

$$-\frac{3e^2\hbar}{4\pi m^2c^2}\frac{1}{\varrho^2}\vec{\varrho}\times\vec{p}-\frac{e^2\hbar^2}{2m^2c^2}\delta^2(\vec{\varrho})$$
$$+\frac{e^2\hbar^2}{2\mu^2m^2c^2}\delta^2(\vec{\varrho})\vec{p}^2 \quad . \tag{34}$$

Except for last term, the second terms with bracket still have some differences from U_M. For the limit $\varrho \to \infty$, U_{MCS} turns out to be

$$U_{MCS}(\vec{p},\vec{\varrho}) \simeq \frac{e^2\hbar}{\pi\mu mc}\frac{1}{\varrho^2}\vec{\varrho}\times\vec{p} \quad . \tag{35}$$

These asymptotic forms of $U_{MCS}(\vec{p},\vec{\varrho})$, Eqs.(34) and (35), are clearly different from $U_M(\vec{p},\vec{\varrho})$ and $U_{CS}(\vec{p},\vec{\varrho})$, but at least, the main structures are similar.

Howevre, if we take the extreme values of $|\mu|$ rather than ϱ, the conclusions are also valid, and the limiting forms of U_{MCS} are almost the same as Eqs.(34) and (35). It is easy to check that the Breit Hamiltonians in momentum space calculated in sections 3.1 and 3.2 are regained, as one takes Eq.(28) in the $|\mu|$ limiting situations before the Fourier transformation is made. Conversely, if the trasformations are made first, then the Breit Hamiltonian in coordinate space in sections 3.1 and 3.2 will never be recovered. All these mean, in brief, integration, *i.e.* Fourier transformation, does not commute with the $|\mu|$ limit-taking operation.

4 Conclusions and Discussions

We have showed that interacting Dirac fermions in $2 + 1$ dimensions behave as anyons in the non-relativistic limit. And, we also discussed briefly the asymptotic behaviors of U_{MCS} with respect to ϱ. From the case of $\mathcal{L}_{MCS}$, one takes a limit $\varrho \to 0(\varrho \to \infty)$, will gains almost the same formulae as the $\mathcal{L}_M$ ($\mathcal{L}_{CS}$) gave. The effects of Maxwell term dominates while the limit, $\varrho \to 0$, is taken. On the other side, if ϱ approaches to infinity, then the Chern-Simons term will dominate. The reason is when global structures are detected, topological properties will emerge explicitly. Since the local observors are always not sensitive to feel the affections of topology, the effects of Maxwell's term will overcome the effects of the topological Chern-Simons term, while two particles are too closed. Oppositely, if two particles are separated far enough, the topological CS's results will suppress the Maxwell's one.

The Hamiltonians (17), (23) and (29) describe the interactions between two identical fermions. However, if one of the fermion is replaced by an antifermion, then, not only the scattering amplitude but also the annihilation amplitude, Fig.2, must take into account [9].

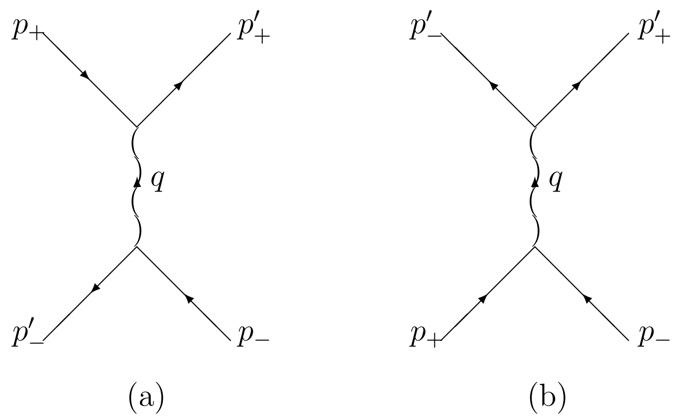

Fig 2. The (a).scattering and (b).annihilation tree diagrams of an electron-positron system. The indices "+" and "−" are defined as positron and electron respectively.

The scattering part is corresponded to the Hamiltonian (17), (23) and (29) with opposite sign in centre-of-mass frame, and the final expression of the annihilation parts are gotten as :

$$\begin{cases} \frac{e^2\hbar^2}{2m^2c^2}\delta^2(\vec{\varrho}) & for \quad \mathcal{L}_M \\ \frac{e^2\hbar^2}{\mu mc}\delta^2(\vec{\varrho}) & for \quad \mathcal{L}_{CS} \quad , \\ \frac{e^2\hbar^2}{4m^2c^2-\mu^2}\left[(2-\frac{\mu}{mc})\delta^2(\vec{\varrho}) - \frac{1}{4m^2c^2}\vec{\nabla}(\delta^2(\vec{\varrho}))\right] & for \quad \mathcal{L}_{MCS} \end{cases} \tag{36}$$

where the result for $\mathcal{L}_M$ enhances the strength of fermions contact term(δ-term), while the outcome for $\mathcal{L}_{CS}$ will cancel the term in pure Chern-Simons's one. Certainly, if $|\mu| \to \infty$, the last annihilation amplitude will give the same result as CS gauge theory. But, as $|\mu| \to 0$, the amplitude does not approach to the result for $\mathcal{L}_M$. This ambiguity is again induced by the same reason for the problem of the asymptotic form of U_{MCS}. It means that, the Fourier transformation does not commute with the limit-taking operation.

APPENDIX

In this appendix, some helpful Fourier transformations in $2+1$ dimension would be exhibited as follows [17]:

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{1}{\vec{q}^2}\frac{d^2\vec{q}}{(2\pi)^2} = -\frac{1}{2\pi}\ln(\varrho) \quad , \tag{37}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{\vec{q}}{\vec{q}^2}\frac{d^2\vec{q}}{(2\pi)^2} = i\frac{1}{2\pi}\frac{\vec{\varrho}}{\varrho^2} \quad , \tag{38}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{(\vec{q}\cdot\vec{a})(\vec{q}\cdot\vec{b})}{\vec{q}^4}\frac{d^2\vec{q}}{(2\pi)^2} = \frac{1}{4\pi}\left[-\ln(\varrho)(\vec{a}\cdot\vec{b}) - \frac{1}{\varrho^2}\vec{\varrho}(\vec{\varrho}\cdot\vec{a})\cdot\vec{b}\right] \quad , \tag{39}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{1}{\vec{q}^2+\mu^2}\frac{d^2\vec{q}}{(2\pi)^2} = \frac{1}{2\pi}K_0(|\mu|\varrho) \quad , \tag{40}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{\vec{q}^2}{\vec{q}^2+\mu^2}\frac{d^2\vec{q}}{(2\pi)^2} = \delta^2(\vec{\varrho}) - \frac{\mu^2}{2\pi}K_0(|\mu|\varrho) \quad , \tag{41}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{\vec{q}}{\vec{q}^2+\mu^2}\frac{d^2\vec{q}}{(2\pi)^2} = i\frac{|\mu|}{2\pi}K_1(|\mu|\varrho)\hat{\varrho} \quad , \tag{42}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}}\frac{1}{\vec{q}^2(\vec{q}^2+\mu^2)}\frac{d^2\vec{q}}{(2\pi)^2} = -\frac{1}{2\pi\mu^2}\left[\ln(\varrho) + K_0(|\mu|\varrho)\right] \quad , \tag{43}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}} \frac{\vec{q}}{\vec{q}^2(\vec{q}^2+\mu^2)} \frac{d^2\vec{q}}{(2\pi)^2} = -i\frac{1}{2\pi|\mu|}K_1(|\mu|\varrho)\hat{\varrho} + i\frac{1}{2\pi\mu^2}\frac{1}{\varrho^2}\vec{\varrho} \quad , \tag{44}$$

$$\int e^{i\vec{q}\cdot\vec{\varrho}} \frac{(\vec{q}\cdot\vec{a})(\vec{q}\cdot\vec{b})}{\vec{q}^2(\vec{q}^2+\mu^2)} \frac{d^2\vec{q}}{(2\pi)^2} = \left[\frac{2}{|\mu|\varrho}K_1(|\mu|\varrho) + K_0(|\mu|\varrho) - \frac{2}{(\mu\varrho)^2}\right] \frac{\vec{\varrho}(\vec{\varrho}\cdot\vec{a})\cdot\vec{b}}{2\pi\varrho^2}$$
$$+ \left[-\frac{1}{|\mu|\varrho}K_1(|\mu|\varrho) + \frac{1}{(\mu\varrho)^2} - \frac{\pi}{\mu^2}\delta^2(\vec{\varrho})\right] \frac{(\vec{a}\cdot\vec{b})}{2\pi} \quad , \tag{45}$$

where $\hat{\varrho}$ is a unit radial vector in a plane. The last equation which we would like to call "dipole" term. Since $\frac{1}{\vec{q}^2(\vec{q}^2+\mu^2)}$ can be separated as $\frac{1}{\mu^2}\left[\frac{1}{\vec{q}^2} - \frac{1}{\vec{q}^2+\mu^2}\right]$, and, for example, first part of the separation is looked as

$$\int e^{i\vec{q}\cdot\vec{\varrho}} \frac{(\vec{a}\cdot\vec{q})\vec{q}}{\vec{q}^2} \frac{d^2\vec{q}}{(2\pi)^2} = -\vec{\nabla}\left[(\vec{a})\cdot\vec{\nabla}(-\frac{1}{2\pi}\ln(\varrho))\right] \quad , \tag{46}$$

which is just the type of dipole electric field with respect to a dipole momentum $\vec{a}$.

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